

A thermodynamic study for the optimization of stable operation of free piston Stirling engines

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Abstract

One of the most novel applications of the Stirling cycle is in the free piston configuration that was initially designed by W. Beale. In free piston Stirling engines (FPSEs), there are no mechanical linkages coupling the pistons or displacers, the motions of the reciprocating components follow the working gas pressure variations. Fillipo de Monte and G. Benvenuto have recently proposed a linearization technique of the dynamic balance equations. The aim of this paper is to predict the thermodynamic conditions for stable operation of FPSEs and their modeling.

The equations of the angular velocity are solved analytically in terms of the working gas mass and the displacer–piston phase angle of the machine. Using the criterion of stable engine cyclic steady operation, a mathematically rigorous form is obtained for the main parameters of the engine.

Furthermore, for simplicity reasons, thermodynamic magnitudes are obtained using the Schmidt analysis (isothermal model).

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1. Introduction

One of the most novel applications of the Stirling cycle is in the free piston configuration that initially was designed by W. Beale at Ohio University in the late 1960s [1,2]. This configuration is the one that holds the most immediate promise. Free piston Stirling engines (FPSEs) have no

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Nomenclature

A	cross-sectional area for moving element (cm^2)
C_C	compression clearance space (mm)
$C_{H_{dc}}$	displacer gas spring damping (N s/m)
$C_{H_{pc}}$	piston gas spring damping (N s/m)
C_{pc}	load damping (N s/m)
D	damping coefficient (s^{-1})
E_E	expansion clearance space (mm)
F	force (N)
K_c	casing stiffness (kg/s^2)
M	mass (g)
n_i	eigenvectors
P	power (kW)
p	pressure (bar)
Q	heat transfer (J/cycle)
R	working gas (helium) constant (J/kg K)
r	displacer–piston amplitude ratio, X_d/X_p
r_i	eigenvalues
S	stiffness coefficient (s^{-2})
T	temperature (K)
t	time (s)
V	volume (cm^3)
v	velocity (m/s)
W	work done (J/cycle)
x	displacement (m)
X	displacement amplitude (m)

Greek symbols

Δ	difference
ϕ	piston–displacer phase angle (degree)
ω	angular velocity (rad/s)

Subscripts

B	bounce space
C	compression space
CL	clearance space
c	casing
d	displacer
E	expansion space
g	gas spring
H	heater
K	cooler

mean	average
p	piston
R	regenerator

Superscripts

·	first order derivative with respect to time
··	second order derivative with respect to time

kinematic mechanism coupling the reciprocating elements to each other or to a common rotating shaft. Instead, the elements move entirely in response to the gas or other spring forces acting upon them. There is a surprising diversity in the various types of FPSEs [3]. The most common arrangement used for useful applications is a single cylinder (b-type).

FPSEs can be used as thermal engines as well as cryocoolers. As prime movers, they use different types of fuel: gas, liquid and solid in some cases. Nuclear [4,5] or solar energy [6] may be utilized also as a power source. This machine usually is connected with a linear alternator (LA) to produce electric power [7].

There are two distinct preferred modes of operation (single cylinder, b-type): (i) Large piston motions relative to casing motions, when the power is mainly from the piston. In this case, the mass of the piston is lighter than that of the casing, and the piston amplitude is greater than the casing amplitude. In this group belongs the RE-1000, which was designed by Sunpower Inc. some years ago. (ii) The other mode is large casing motions relative to piston motions, when the power is mainly derived from the casing, and its mass is lighter than that of the piston. Respectively, the casing amplitude is greater than the piston amplitude. Such type of machine is the M-100, which was also designed by Sunpower Inc.

The main advantages of FPSEs relative to kinematic Stirling engines are: (i) simpler mechanical design; (ii) the possibility to avoid any side force on the piston and also, by using flexure bearings, the possibility to eliminate all rubbing parts, reducing engine wear [8]; (iii) a high energy conversion (thermal to mechanical) efficiency; (iv) easy starting; (v) high performance; (vi) very long life; and (vii) low cost. The principal disadvantage of the FPSE is simply the lack of a rotating shaft. So many machines in engineering involve rotating shafts that their absence in FPSEs is sometimes regarded as a disadvantage, although in practice, the disadvantage is not so great as it appears. Many systems involving pumps, compressors and other machinery include kinematic mechanisms whereby the rotating motion of driven shafts is converted to reciprocating motion. All these can be directly driven by FPSEs and so can a reciprocating pump providing high pressure hydraulic fluid to drive hydraulic motors (e.g. in vehicles) [3].

The correct choice of piston, displacer and casing masses, gas springs and other components are necessary for stable dynamic behavior. On all FPSEs, the displacer is typically much lighter than the piston. For positive power, the piston and casing motion are required to lag behind that of the displacer. FPSEs have load dependent frequencies. However, it will be shown that in some designs, with prudent selection of geometry, it is possible to reduce the variation of frequency with load to a negligible level [9].

In most FPSEs, the gas flow is predominantly turbulent in the cooler and heater and laminar in the regenerator. The pressure drop in the machine is the summation of the pressure drops in the

cooler, regenerator and heater. However, the variation of density with pressure is neglected, since in most high pressure FPSEs, the pressure swing is usually a small fraction of the charge pressure, typically less than 15%. It is, therefore, assumed that density is only a function of local temperature, which, in the isothermal case, is constant.

Optimizations of the dynamic behavior of FPSEs are not a simple matter because of the lack of mechanical linkages that are able to fix strokes (displacer and piston) and phase angle for the moving elements (displacer and piston).

Because the laws of motion of the moving elements (displacer and piston) of FPSEs are not known presumptively, as in the case of kinematic Stirling engines, they have to be calculated by means of an accurate dynamic analysis. If the equations of motion are linearized (linear dynamic analysis), there will be a system of linear and homogeneous differential equations with constant coefficients. These equations may be solved analytically in the time domain. Solution of the polynomial characteristic equation yields the condition for periodic steady operation of the FPSEs [10].

An attempt has been by Rogdakis et al. [11], in which the equations of motion have been solved analytically in terms of stiffness and damping coefficients.

2. Analysis

2.1. Generalized analysis

In order to describe the action of FPSEs, it is useful to assume harmonic motion of the components, which is often nearly true. The mass-spring-damper system in Fig. 1 illustrates the statements made above. The damping force represents work done by the mass, and as result of this, the applied force $\mathbf{F}$ must have a component normal to the displacing surface (parallel to the velocity) pointing up. These three forces (the applied force, the force due to the spring effect and the force due to the damping effect) are collinear, as it seems in the following equation [12].

Newton's second law yields:

$$F + F_{\text{spring}} + F_{\text{damping}} = M \cdot \ddot{x} \quad (1)$$

In a simple FPSE, there are three oscillating components, the piston, displacer and casing. If S_{ij} and D_{ij} account for the influence on the i -component motions by virtue of a spring and damping, respectively, coupling to the motions of the j -component, whereas S_{ii} or D_{ii} are a spring or damping effect unique to the i -component, respectively, the forms of the dynamic equations of the component parts (per unit element mass) are generalized as follows:

$$\ddot{x}_p = S_{pp}x_p + S_{pd}x_d + S_{pc}x_c + D_{pp}\dot{x}_p + D_{pd}\dot{x}_d + D_{pc}\dot{x}_c \quad (2)$$

$$\ddot{x}_d = S_{dp}x_p + S_{dd}x_d + S_{dc}x_c + D_{dp}\dot{x}_p + D_{dd}\dot{x}_d + D_{dc}\dot{x}_c \quad (3)$$

$$\ddot{x}_c = S_{cp}x_p + S_{cd}x_d + S_{cc}x_c + D_{cp}\dot{x}_p + D_{cd}\dot{x}_d + D_{cc}\dot{x}_c \quad (4)$$

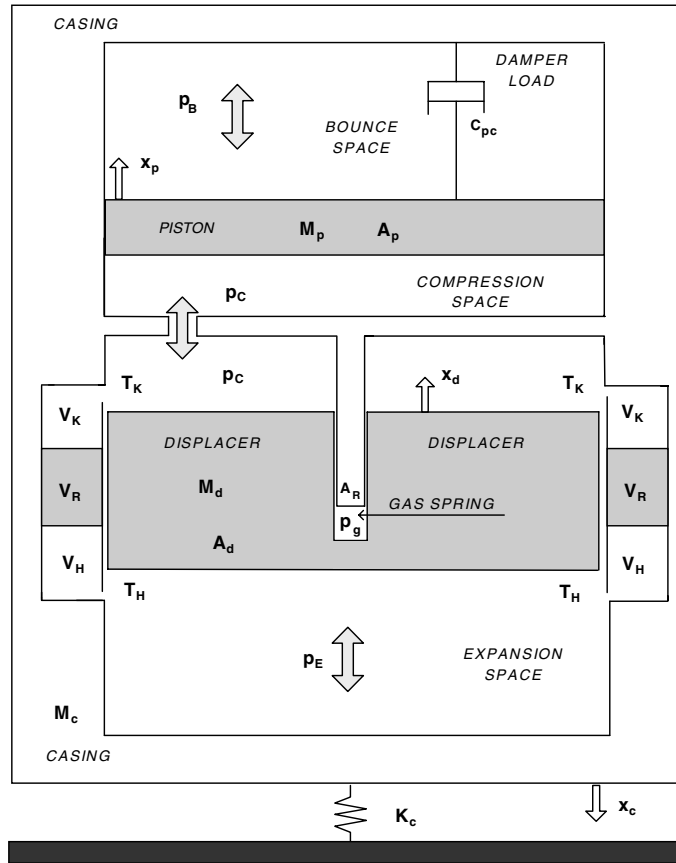


Fig. 1. Notation of a typical FPSE type (i)/light mass piston.

The above equation system may also be written in matrix form:

$$\begin{bmatrix} \ddot{x}_p \\ \ddot{x}_d \\ \ddot{x}_c \end{bmatrix} = \begin{bmatrix} S_{pp} & S_{pd} & S_{pc} \\ S_{dp} & S_{dd} & S_{dc} \\ S_{cp} & S_{cd} & S_{cc} \end{bmatrix} \cdot \begin{bmatrix} x_p \\ x_d \\ x_c \end{bmatrix} + \begin{bmatrix} D_{pp} & D_{pd} & D_{pc} \\ D_{dp} & D_{dd} & D_{dc} \\ D_{cp} & D_{cd} & D_{cc} \end{bmatrix} \cdot \begin{bmatrix} \dot{x}_p \\ \dot{x}_d \\ \dot{x}_c \end{bmatrix} \quad (5)$$

In the case of a FPSE of the b-type, the equation system using thermodynamic magnitudes can be written:

For the piston:

$$M_p \ddot{x}_p = (p_c - p_B) A_p - (C_{H_{pc}} + C_{pc})(\dot{x}_p + \dot{x}_c) \quad (6)$$

For the displacer:

$$M_d \ddot{x}_d = A_d \Delta p + A_R (p_c - p_d) - C_{H_{dc}} (\dot{x}_d + \dot{x}_c) \quad (7)$$

For the casing:

$$M_c \ddot{x}_c = A_d (p_c + \Delta p - p_B) + A_R (p_c - p_d) - (C_{pc} + C_{H_{pc}})(\dot{x}_c + \dot{x}_p) - K_c x_c \quad (8)$$

where the expansion space pressure is taken to be given by

$$p_E = \Delta p + p_C \quad (9)$$

Because of the predominantly piston motion mode, the casing motion is ignored, and the describing equations are:

For the piston:

$$M_p \ddot{x}_p = (p_c - p_B)A_p - (C_{H_{pc}} + C_{pc})\dot{x}_p \quad (10)$$

For the displacer:

$$M_d \ddot{x}_d = A_d \Delta p + A_R(p_c - p_d) - C_{H_{dc}}\dot{x}_d \quad (11)$$

The operation of the engine is described by the isothermal analysis (Schmidt analysis) [13]. Using linear dynamics theory [9] that adequately describes the behavior of the FPSE, the spring and damping coefficients of the above equations as functions of the working gas mass M , angular velocity ω and piston–displacer phase angle ϕ , which is the phase angle between the motion of the piston and the motion of the displacer, are:

$$S_{pp}(M) = -\frac{A_p^2}{M_p} p_{\text{mean}}(M) \left(\frac{1}{T_K S} + \frac{\gamma}{V_B} \right) \quad (12)$$

$$S_{pd}(M) = -\frac{A_p^2}{M_p S} p_{\text{mean}}(M) \left[(1/T_H) - \left(1 - \frac{A_R}{A_p} \right) / T_K \right] \quad (13)$$

$$S_{dp}(M) = -\frac{A_p A_R p_{\text{mean}}(M)}{M_D T_K S} \quad (14)$$

$$S_{dd}(M) = -\frac{A_R p_{\text{mean}}(M)}{M_D} \left[\frac{A_p}{T_H S} - \frac{(A_p - A_R)}{T_K S} + \gamma \frac{A_R}{V_D} \right] \quad (15)$$

$$D_{pp} = -\frac{C_{pc}}{M_p} \quad (16)$$

$$D_{pd} = 0 \quad (17)$$

$$D_{dp}(M, \omega, \phi) = \frac{C_p(M, \omega, \phi)}{M_D} \quad (18)$$

$$D_{dd}(M, \omega, \phi) = \frac{C_d(M, \omega, \phi) - C_{H_{dc}}}{M_D} \quad (19)$$

where

$$S = \frac{A_p C_C}{T_K} + \frac{V_K}{T_K} + \frac{V_R \ln[T_H/T_K]}{T_H - T_K} + \frac{V_H}{T_H} + \frac{A_d E_E}{T_H} \quad (20)$$

Now, the equation system (5) can be represented as follows:

$$\dot{\mathbf{x}} = \mathbf{A} \cdot \mathbf{x} \quad (21)$$

where

$$\dot{\mathbf{x}} = \begin{bmatrix} \ddot{x}_p \\ \ddot{x}_d \\ \dot{x}_p \\ \dot{x}_d \end{bmatrix} \quad (22)$$

$$\mathbf{A} = \begin{bmatrix} D_{pp} & D_{pd} & S_{pp} & S_{pd} \\ D_{dp} & D_{dd} & S_{dp} & S_{dd} \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \quad (23)$$

and

$$\mathbf{x} = \begin{bmatrix} \dot{x}_p \\ \dot{x}_d \\ x_p \\ x_d \end{bmatrix} \quad (24)$$

The solution of the above system of differential equations is based on the roots of the characteristic polynomial:

$$G(r) = r^4 + a \cdot r^3 + \beta \cdot r^2 + \gamma \cdot r + \delta \quad (25)$$

where

$$\alpha = D_{dd}(M, \omega, \phi) + D_{pp} \quad (26)$$

$$\beta = D_{pp} \cdot D_{dd}(M, \omega, \phi) - D_{pd} \cdot D_{dp}(M, \omega, \phi) - S_{dd}(M) - S_{pp}(M) \quad (27)$$

$$\gamma = D_{dp}(M, \omega, \phi) \cdot S_{pd}(M) + S_{dp}(M) \cdot D_{pd} - D_{dd}(M, \omega, \phi) \cdot S_{pp}(M) - S_{dd}(M) \cdot D_{pp} \quad (28)$$

and

$$\delta = S_{pp}(M) \cdot S_{dd}(M) - S_{pd}(M) \cdot S_{dp}(M) \quad (29)$$

The eigenvalues of $\mathbf{A}$ are r_1, r_2, r_3 and r_4 , with corresponding eigenvectors $\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3$ and $\mathbf{n}_4$. Therefore, the general solution of Eq. (20) is

$$\mathbf{x}(t) = c_1 \cdot e^{r_1 \cdot t} \cdot \mathbf{n}_1 + c_2 \cdot e^{r_2 \cdot t} \cdot \mathbf{n}_2 + c_3 \cdot e^{r_3 \cdot t} \cdot \mathbf{n}_3 + c_4 \cdot e^{r_4 \cdot t} \cdot \mathbf{n}_4 \quad (30)$$

Because the vectors $\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3$ and $\mathbf{n}_4$ are linearly independent, it is possible to find a unique set of constants c_1, c_2, c_3 and c_4 to satisfy any initial condition. The initial values $\mathbf{x}(0)$ imply that

$$\mathbf{x}(0) = c_1 \cdot \mathbf{n}_1 + c_2 \cdot \mathbf{n}_2 + c_3 \cdot \mathbf{n}_3 + c_4 \cdot \mathbf{n}_4 \quad (31)$$

According to the form of these roots, we have the following cases:

- (a) *Roots with positive real parts:* The exponential functions tend to infinity with time, and there is not any oscillation.
- (b) *Roots with only negative real part:* The equations of motion of the piston and displacer tend to zero. If there are two complex conjugate roots, there are convergent oscillations.

- (c) *Two roots with negative real parts and two roots with positive real part:* $x_p(t)$ and $x_d(t)$ increase with the time. If the roots with positive real parts are complex conjugates, there is an oscillation for each component, although they are divergent.
- (d) *Two imaginary roots and two roots with a negative real part:* Steady oscillations.

The main conclusion is that the machine works at a cyclic steady state when the characteristic polynomial has two imaginary roots and two roots with a negative real part. The absolute values of this real part must be very high to ensure a quick drop of the transient and, therefore, a quick arrival of the stationary oscillations for the reciprocating parts.

For secure stable operation [10,14] of the engine, the roots of the characteristic polynomial (Eq. (25)) should be

$$r_1 = k \cdot i, \quad r_2 = -k \cdot i, \quad r_3 = l \quad \text{and} \quad r_4 = m \quad (32)$$

According to the de Moivre form, these should effect

$$(r - k \cdot i)(r + k \cdot i)(r - l)(r - m) = 0 \Rightarrow r^4 - (m + l)r^3 + (ml + k^2)r^2 - k^2(m + l)r + k^2ml = 0 \quad (33)$$

The above polynomial (Eq. (33)) must be equal to the characteristic polynomial $G(r)$, (Eq. (25)). From this requirement, the following equations are obtained:

$$-(m + l) = \alpha \quad (34)$$

$$ml + k^2 = \beta \quad (35)$$

$$-k^2(m + l) = \gamma \quad (36)$$

$$k^2ml = \delta \quad (37)$$

From Eqs. (34) and (36), we obtain the magnitude of the imaginary roots that expresses the angular velocity ω :

$$\omega^2 = k^2 = \frac{\gamma}{\alpha} \quad (38)$$

The angular velocity ω as a function of S_{ii} and D_{ij} is given in the following equation:

$$\omega = \left[\frac{D_{dp}(M, \omega, \phi) \cdot S_{pd}(M) + S_{dp}(M) \cdot D_{pd} - D_{dd}(M, \omega, \phi) \cdot S_{pp}(M) - S_{dd}(M) \cdot D_{pp}}{D_{dd}(M, \omega, \phi) + D_{pp}} \right]^{1/2} \quad (39)$$

Using Eqs. (37) and (38), Eq. (35) is written:

$$\beta = \frac{\gamma}{\alpha} + \frac{\delta\alpha}{\gamma} \Rightarrow \left(\frac{\gamma}{\alpha}\right)^2 - \beta\left(\frac{\gamma}{\alpha}\right) + \delta = 0 \quad (40)$$

Substituting $\frac{\gamma}{\alpha}$ as ω^2 , the geometric constraint is obtained [9]:

$$\omega^4 - \beta\omega^2 + \delta = 0 \quad (41)$$

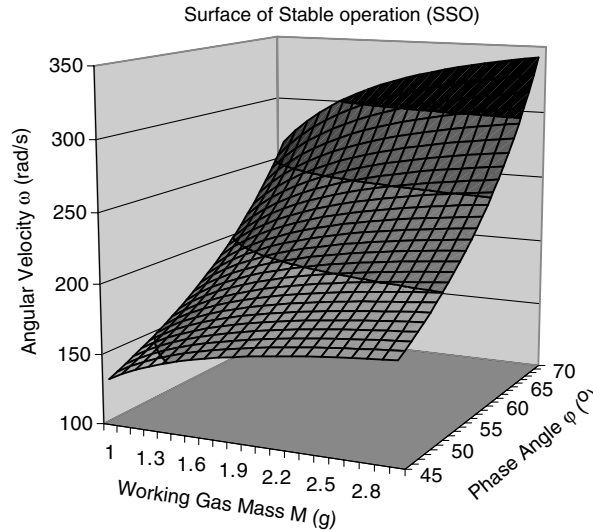


Fig. 2. Surface of stable operation (SSO) of RE-1000.

The solution of the biquadratic equation using the coefficients S_{ii} and D_{ij} is

$$\omega = \left[\frac{\beta - (\beta^2 - 4\delta)^{1/2}}{2} \right]^{1/2} \quad (42)$$

Furthermore, the parameters α , β , γ and δ should be greater than zero, something that is usually satisfied. Eqs. (39) and (42) should have equal solutions, which is the acceptable angular velocity for stable operation of the FPSE.

2.1.1. Case study I

The previous analysis has been applied to a typical FPSE, the RE-1000 produced by Sunpower Inc. The diagram of angular velocity versus the displacer–piston amplitude phase angle and the working gas mass follows.

As shown in Fig. 2, the effect of the phase angle on the angular velocity is much stronger than the effect of the working gas mass. This could be explained from the fact that the phase angle and angular velocity have effects on the same terms of the damping coefficients, while the working gas mass, except these, have effects on the stiffness coefficients. The diagram surface shows the stable operation area in a range from 0.5 to 3 gr working gas mass and from 50° to 75° phase angle.

- (i) In the area below the surface of stable operation (SSO), both the piston and displacer motions are unsteady and oscillate with increasing amplitudes.
- (ii) In the area above the SSO, the piston and displacer motions are oscillations whose amplitudes tend to zero.

In Figs. 3 and 4, the stable operations of the engine are shown for a point on the SSO.

In Figs. 5 and 6, the unstable operations of the engine are shown (case (b)—roots with only negative real parts—oscillations whose amplitudes tend to zero—stoppage). The greater the

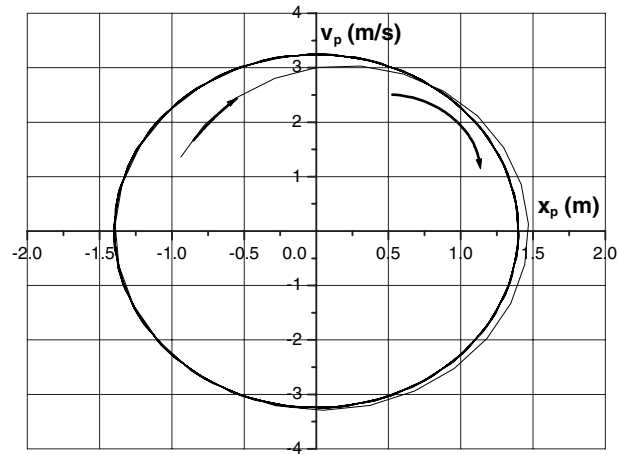


Fig. 3. Stable operation. Piston velocity V_p versus piston amplitude X_p .

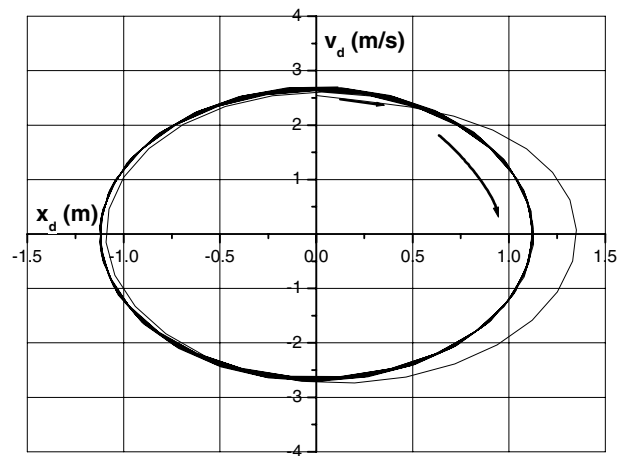


Fig. 4. Stable operation. Displacer velocity V_d versus displacer amplitude X_d .

angular velocity is than the optimum (which is given by the SSO), the faster the displacer and piston amplitudes are reduced.

In Figs. 7 and 8, the unstable operations of the engine are shown, (case (c)—roots with positive real parts are complex conjugates—there is an oscillation for each component, although they are divergent). Respectively, in this case, the less the angular velocity is than the optimum (which is given by the SSO), the faster the displacer and piston amplitudes increase.

2.1.2. Case study II

Typically, the operating values for the thermodynamic parameters of this engine, in which the working gas is helium, are 814 K heater temperature and 322 K cooler temperature. The above

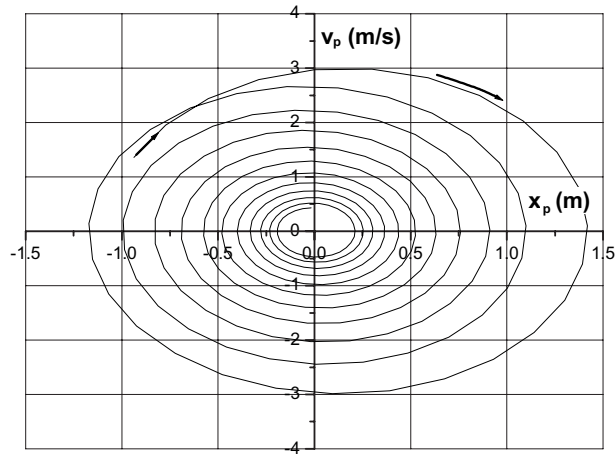


Fig. 5. Unstable operation (stoppage). Piston velocity V_p versus piston amplitude X_p .

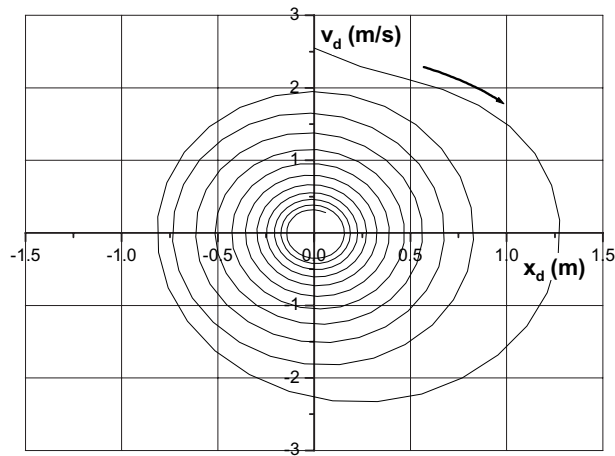


Fig. 6. Unstable operation (stoppage). Displacer velocity V_d versus displacer amplitude X_d .

analysis has been used while keeping these parameters constant, and the differential equations system has been solved parametrically for two modes.

1. On the first mode, the independent parameter is the working gas mass, and the dependent parameters are the phase angle and the angular velocity.
2. On the second mode, the independent parameter is the phase angle, and the dependent parameters are the working gas mass and the angular velocity.

The following Figs. 9–12 show how the working gas mass affects each operation parameter and other magnitudes, such as work done and indicated power output.

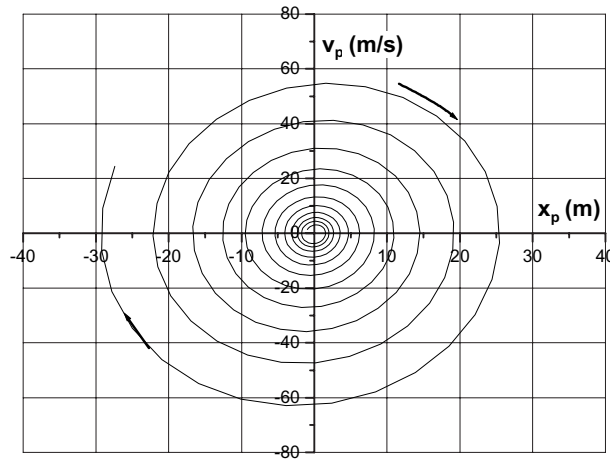


Fig. 7. Unstable operation (oscillation with increasing amplitude). Piston velocity V_p versus piston amplitude X_p .

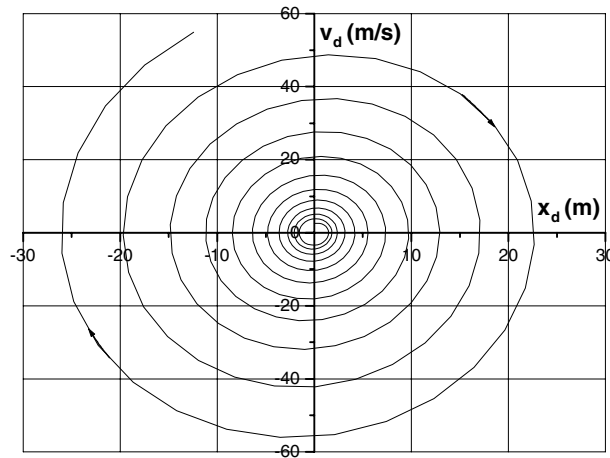


Fig. 8. Unstable operation (oscillation with increasing amplitude). Displacer velocity V_d versus displacer amplitude X_d .

In Fig. 9, the angular velocity versus the working gas mass is shown. On the solid line, there are the results of mode 1. The dot lines are the results of case study I and represent stable operation points for constant values of the phase angle. The points at which the solid line cuts the dot lines are the points at which mode 1 gives the same results as case study I. The solid line is represented approximately by the following equation:

$$\omega = 158.2 \cdot M^{0.55} \quad (43)$$

Fig. 10 shows how the piston and displacer amplitudes vary with the working gas mass. Also there is the ratio r of the amplitude X_d to the amplitude X_p . As shown, when the working gas mass is lower than 0.82 g, the piston and displacer amplitudes are constant, and the displacer amplitude

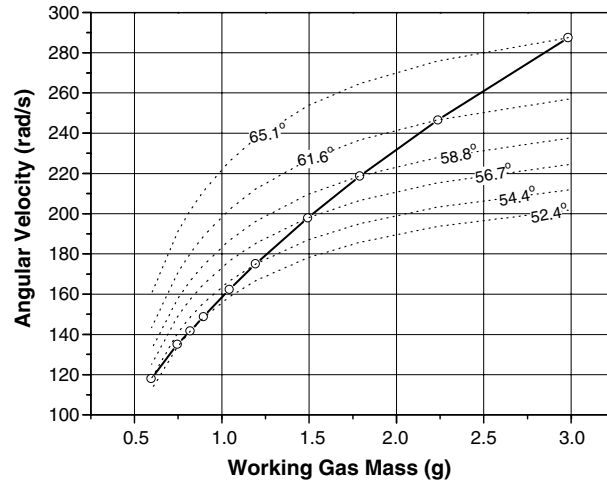


Fig. 9. Angular velocity versus working gas mass (for different phase angle values).

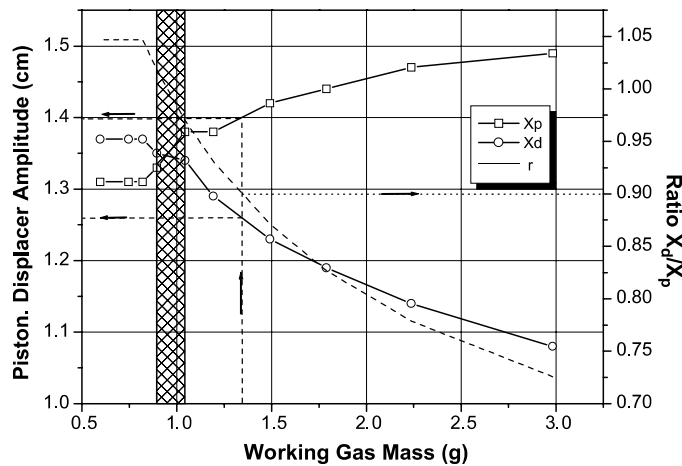


Fig. 10. Displacer amplitude, piston amplitude and ratio X_d/X_p versus working gas mass.

is greater. For a working gas mass greater than 0.82 g, the displacer amplitude decreases and the piston amplitude increases. After the value of 1 g, it is clear that the piston amplitude becomes greater than that of the displacer. The ratio r is also constant in the beginning and then decreases according to the following polynomial equation:

$$r = 0.81 + 0.909 \cdot M - 1.01 \cdot M^2 + 0.284M^3 \quad (44)$$

For calculations of the mean pressure of the engine, work done, heat transfer in cold and hot spaces and indicated power output, the isothermal analysis is used. Fig. 11 shows the mean pressure versus working gas mass and angular velocity. The mean pressure increases as the

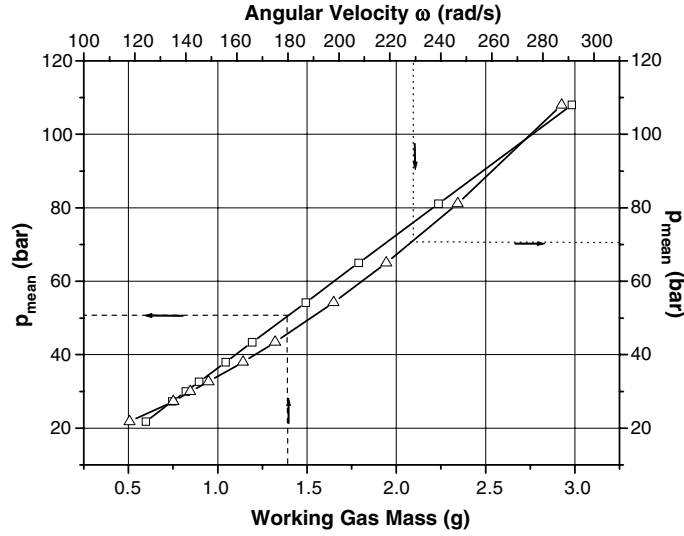


Fig. 11. Mean pressure versus working gas mass and versus angular velocity.

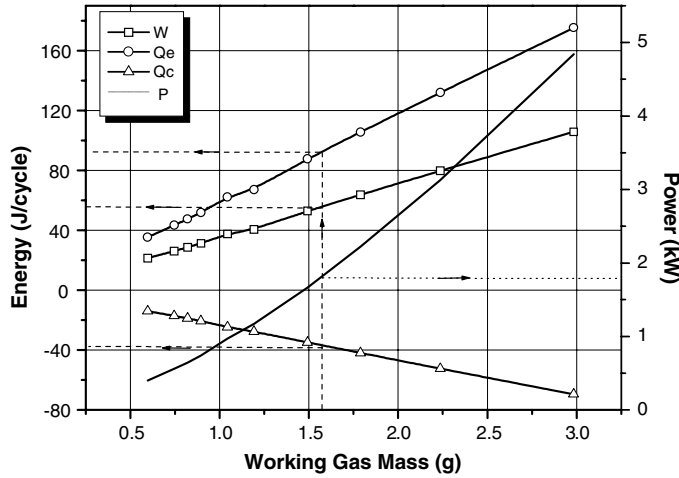


Fig. 12. Energy transfers and power versus working gas mass.

working gas mass increases, while it decreases as the angular velocity decreases. The equations, which describe the mean pressure variation, follow:

$$p_{\text{mean}} = 36.3 \cdot M \quad (45)$$

$$p_{\text{mean}} = 1.13 + 0.038 \cdot \omega + 0.0012 \cdot \omega^2 \quad (46)$$

The work done per cycle is equal to the sum of the heat transfer to the gas in the hot space per cycle and the heat transfer from the gas in the cold space per cycle. The work done and heat

transfers are shown in Fig. 12. On the right axis of the diagram, there is the indicated power output. The indicated power output increases as the working gas mass increases. The following equations represent the above magnitudes.

$$W = 35.4 \cdot M \quad (47)$$

$$Q_e = 58.7 \cdot M \quad (48)$$

$$Q_c = -23.3 \cdot M \quad (49)$$

$$P = 0.546 \cdot M + 0.367 \cdot M^2 \quad (50)$$

In contrast to mode 1, in mode 2, there are two acceptable values of each magnitude for each value of the phase angle. There is a minimum phase angle equal to the critical phase angle of 52.4° with critical angular velocity 140.3 rad/s, below which mode 2 does not find any solution for stable operation. For better comprehension, the results of this mode are separated into two groups. In Group 1 (G1), the values of the magnitudes that correspond to the values of the angular velocity greater than the critical are included. In Group 2 (G2), the values of the magnitudes that correspond to the values of the angular velocity less than the critical are included.

The following Figs. 13–16 show how the phase angle affects each operation parameter and other magnitudes, such as the work done and the indicated power output.

In Fig. 13, the angular velocity and working gas mass versus the phase angle are shown. As shown in the diagram, the angular velocity in G1 increases at a greater rate than it decreases in G2. The equations for each group for angular velocity and working gas mass are the following.

$$G1 : \quad \omega = -4867.1 + 231.6 \cdot \phi - 3.6 \cdot \phi^2 + 0.02 \cdot \phi^3 \quad (51)$$

$$G2 : \quad \omega = 26808.1 - 1633.1 \cdot \phi + 37.4 \cdot \phi^2 - 0.4 \cdot \phi^3 + 1.4 \times 10^{-3} \cdot \phi^4 \quad (52)$$

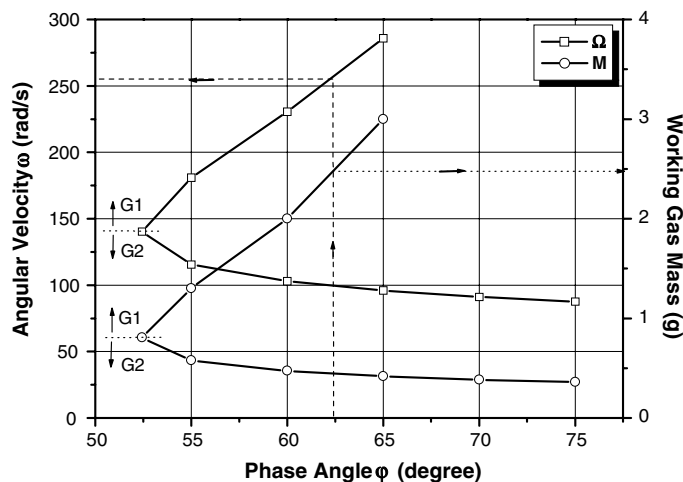


Fig. 13. Angular velocity and working gas mass versus phase angle.

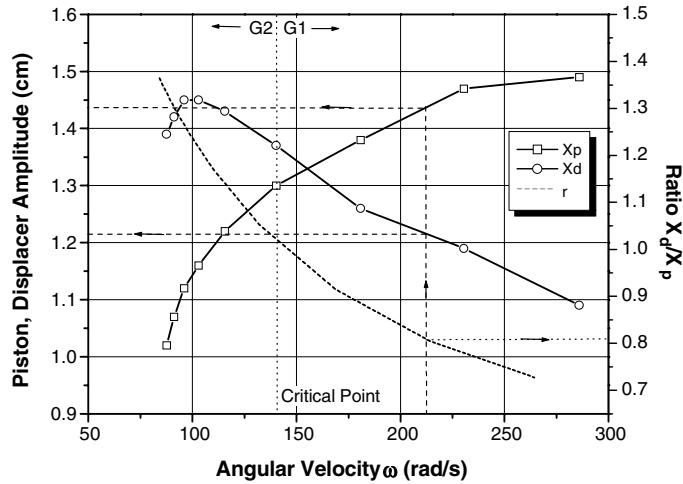


Fig. 14. Displacer amplitude, piston amplitude and ratio X_d/X_p versus angular velocity.

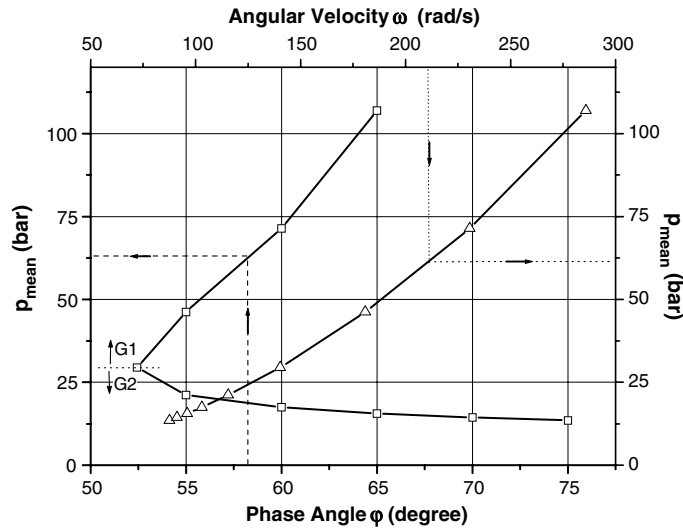


Fig. 15. Mean pressure versus phase angle and versus angular velocity.

$$G1: M = -107.5 + 5.4 \cdot \phi - 9.1 \times 10^{-2} \cdot \phi^2 - 5.2 \times 10^{-4} \cdot \phi^3 \quad (53)$$

$$G2: M = (-141.4 + 2.7 \cdot \phi)^{-0.243} \quad (54)$$

In Fig. 14, the piston and displacer amplitudes and ratio r of amplitude X_d to amplitude X_p are shown. Although the piston amplitude increases with angular velocity, the displacer amplitude has a maximum of about 1.45 cm at 103 rad/s. The ratio r is reduced continuously, beginning greater than 1.0 (displacer amplitude greater than piston amplitude), becoming equal to 1.0 at

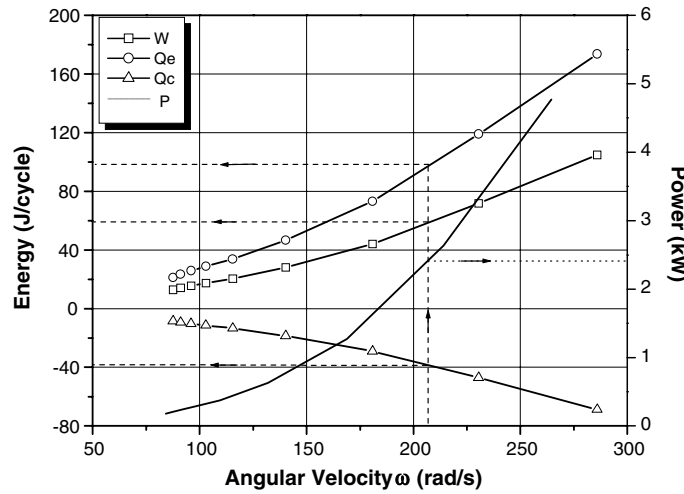


Fig. 16. Energies (W , Q_e , Q_c) and power versus angular velocity.

145 rad/s and continuing reducing. The two groups are separated with the dot line, as shown in the diagram. The relevant equation of the ratio r is the following:

$$r = 2.12 - 0.011 \cdot \omega + 3.02 \times 10^{-5} \cdot \omega^2 - 2.83 \times 10^{-8} \cdot \omega^3 \quad (55)$$

As shown in Fig. 15, the area of high values of angular velocity corresponds to high values of mean pressure. In Fig. 15, it is clear that the mean pressure increases as the angular velocity increases. The equations, which describe the mean pressure variation, are the following:

$$G1 : p_{\text{mean}} = -3868.8 + 194.6 \cdot \phi - 3.3 \cdot \phi^2 + 0.02 \cdot \phi^3 \quad (56)$$

$$G2 : p_{\text{mean}} = \frac{-14061.1 + 279.3 \cdot \phi}{1 - 14.8 \cdot \phi + 0.29 \cdot \phi^2} \quad (57)$$

$$p_{\text{mean}} = 0.42 + 0.055 \cdot \omega + 0.001 \cdot \omega^2 + 1.68 \cdot \omega^3 \quad (58)$$

In Fig. 16, the work done per cycle, heat transfers in the hot and cold spaces per cycle and the indicated power output are shown. The following equations represent the above magnitudes:

$$W = 2.65 + 0.008 \cdot \omega + 0.001 \cdot \omega^2 - 3.05 \times 10^{-7} \cdot \omega^3 \quad (59)$$

$$Q_e = 4.34 + 0.013 \cdot \omega + 0.002 \cdot \omega^2 - 5.05 \times 10^{-7} \cdot \omega^3 \quad (60)$$

$$Q_c = -1.74 - 0.005 \cdot \omega - 8.6 \times 10^{-4} \cdot \omega^2 + 2 \times 10^{-7} \cdot \omega^3 \quad (61)$$

$$P = 0.084 - 1.6 \times 10^{-3} \cdot \omega + 1.68 \cdot \omega^2 + 1.61 \cdot \omega^3 \quad (62)$$

In Table 1, the main parameters of the RE-1000 are shown, as it has been evaluated from Urieli and Berchowitz [9], Walker and Senft [3], the authors' previous analysis [11], the two case studies of the present analysis and experiments [9]. All results seem to compare well with the phase angle, slightly better in case studies 1 and 2. The comparison with experimental results in the amplitude

Table 1
Comparison of the operation parameters of RE-1000

Parameters	Urieli and Berch.	Walker and Senft	Previous analysis [11]	Case study 1	Case study 2	Experiment
Frequency (Hz)	33.2	32.9	32.9	30.0	30.1	30.0
Phase angle (°)	–57.9	–55.1	–55.1	–54.8	–55	–42.5
Amplitude ratio	0.62	0.63	0.63	0.87	0.73	1.06

ratio is quite better in the present analysis, more specifically in case study 1. The same appears in the comparison of frequency, in which we can obtain the exact experimental value using the case study 1.

3. Conclusions

- The stiffness coefficients S_{ij} are dependent only on the working gas mass M and not on the phase angle ϕ or angular velocity ω . For this reason, the constant order δ , of the characteristic equation of the differential equations system, is also dependent only on the working gas mass M .
- The machine works at a cyclic steady state when the characteristic polynomial has two imaginary roots and two roots with negative real parts. The absolute values of these real parts must be very high to ensure a quick drop of the transient and, therefore, a quick arrival of the stationary oscillations for the reciprocating parts. The farther an operation point is from the SSO, the faster the operation of the engine will be increased or decreased in each case.
- The result of case study I is a surface of stable operation (SSO). In the area below the SSO, both the piston and displacer motions are unsteady and oscillate with increasing amplitudes. In the area above the SSO, the piston and displacer motions are oscillations whose amplitudes tend to zero.
- As it seems in Fig. 8, the results of mode 1 of case study II are points of the SSO of case study I. The results of mode 2 are also points of the SSO.
- In mode 1 of case study II, the piston and displacer amplitude are constant until 0.82 g working gas mass, with the piston amplitude greater. After this value, the piston amplitude is decreased, the displacer amplitude is increased, and as a result, the ratio r is decreased from 1.04 to 0.76. The ratio r becomes one when the working gas mass is approximately 0.9 g.
- In contrast to mode 1, in mode 2, there are two acceptable values of each magnitude for each value of the phase angle. There is a minimum phase angle equal to the critical phase angle of 52.4° with critical angular velocity 140.3 rad/s, below which mode 2 does not find any solution for stable operation.
- The variation of mean pressure is greater in Group 1 of mode 2 than in Group 2, as is clearly seen in Fig. 14. The same behavior follows the work done per cycle and the power.

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